

On the Generalized Burnside Theorem

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Abstract The Generalized Burnside Theorem, due to Laudal [2], generalizes the classical Burnside Theorem and is obtained using noncommutative deformations of the family of simple right A -modules when A is a finite dimensional associative algebra over an algebraically closed field. In this paper, we prove a form of the Generalized Burnside Theorem that is more general, where we do not assume that k is algebraically closed. The main purpose of the paper is to clarify and generalize the proof. As an application of the theorem, we introduce a standard form for finite dimensional algebras.

1 Introduction

Let k be a field, let A be a finite dimensional associative algebra over k , and let $\mathbf{M} = \{M_1, \dots, M_r\}$ be the family of simple right A -modules, up to isomorphism. We consider the algebra homomorphism

$$\rho : A \rightarrow \bigoplus_{i=1}^r \text{End}_k(M_i)$$

given by right multiplication of A on the family \mathbf{M} . By the extended version of the classical Burnside Theorem, ρ is surjective when k is algebraically closed. In fact, Artin-Wedderburn theory gives a version of the theorem that holds over any field:

Theorem (Classical Burnside Theorem) *If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $\rho : A \rightarrow \bigoplus_i \text{End}_k(M_i)$ is surjective.*

If A is not a semisimple algebra, then ρ is not injective, since $\ker(\rho) = J(A)$ is the Jacobson radical of A . To improve upon this, we consider the noncommutative deformation functor $\text{Def}_{\mathbf{M}}$ of the family $\mathbf{M} = \{M_1, \dots, M_r\}$ of right A -modules,

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with pro-representing hull H and versal family M_H . There is an induced algebra homomorphism η making the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{\eta} & (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \\ & \searrow \rho & \downarrow \\ & & \bigoplus_{i=1}^r \text{End}_k(M_i) \end{array}$$

The algebra $\mathcal{O}(\mathbf{M}) = \text{End}_H(M_H) \cong (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j))$ is called the algebra of observables, and the map $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$, given by right multiplication of A on the versal family M_H , is called the versal morphism. By Laudal [2], η is an isomorphism when k is algebraically closed. In this paper, we prove a more general version of this result:

Theorem (Generalized Burnside Theorem) *The versal morphism $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$ is injective, and if $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then η is an isomorphism. In particular, η is an isomorphism if k is algebraically closed.*

In case $D_i = \text{End}_A(M_i)$ is a division algebra with $\dim_k D_i > 1$ for some simple module M_i , it is often not difficult to describe the image of η as a subalgebra of $\mathcal{O}(\mathbf{M})$, and we shall give examples. As an application of the theorem, we introduce the standard form of any finite dimensional algebra A , given as

$$A \cong \mathcal{O}(\mathbf{M}) = (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j))$$

when $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, or as a subalgebra of $\mathcal{O}(\mathbf{M})$ in general. Finally, we prove the closure property of the algebra of observables:

Theorem (Closure Property) *Let A be a finitely generated associative k -algebra, and let $\mathbf{M} = \{M_1, \dots, M_r\}$ be any family of finite dimensional right A -modules. If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then the induced algebra homomorphism*

$$\eta^B : B \rightarrow \mathcal{O}^B(\mathbf{M}) \text{ with } B = \mathcal{O}(\mathbf{M})$$

is an isomorphism.

2 Noncommutative deformations of modules

Let A be an associative algebra over a field k . For any right A -module M , there is a deformation functor $\text{Def}_M : \mathbf{l} \rightarrow \mathbf{Sets}$ defined on the category \mathbf{l} of commutative Artinian local k -algebras with residue field k . Deformations in $\text{Def}_M(R)$ are called *commutative deformations* since the base ring R is commutative.

Noncommutative deformations were introduced in Laudal [2]. The deformations considered by Laudal are defined over certain noncommutative base rings instead of the commutative base rings in \mathbf{l} . In what follows, we shall give a brief account of noncommutative deformations of modules. We refer to Laudal [2], Eriksen [1] for further details.

For any positive integer r and any family $\mathbf{M} = \{M_1, \dots, M_r\}$ of right A -modules, there is a *noncommutative deformation functor* $\text{Def}_{\mathbf{M}} : \mathbf{a}_r \rightarrow \mathbf{Sets}$, defined on the category \mathbf{a}_r of noncommutative Artinian r -pointed k -algebras with exactly r simple

modules (up to isomorphism). We recall that an r -pointed k -algebra R is one fitting into a diagram of rings $k^r \rightarrow R \rightarrow k^r$, where the composition is the identity. The condition that R has exactly r simple modules holds if and only if $\bar{R} \cong k^r$, where $\bar{R} = R/J(R)$ and $J(R)$ denotes the Jacobson radical of R .

The noncommutative deformations in $\text{Def}_M(R)$ are equivalence classes of pairs (M_R, τ_R) , where M_R is an R -flat R - A bimodule on which k acts centrally, and $\tau_R : k^r \otimes_R M_R \rightarrow M$ is an isomorphism of right A -modules with $M = M_1 \oplus \cdots \oplus M_r$. In concrete terms, an algebra R in \mathfrak{a}_r is a matrix ring $R = (R_{ij})$ with $R_{ij} = e_i R e_j$. As left R -modules, we have that $M_R \cong (R_{ij} \otimes_k M_j)$ and its right A -module structure is given by an algebra homomorphism

$$\eta_R : A \rightarrow \text{End}_R(M_R) \cong (R_{ij} \otimes_k \text{Hom}_k(M_i, M_j))$$

that lifts $\rho : A \rightarrow \bigoplus_i \text{End}_k(M_i)$. Explicitly, we interpret $\eta_R(a)$ as a right action of a via

$$\eta_R(a) = \sum_i e_i \otimes \rho_i + \sum_{i,j,l} r_{ij}^l \otimes \phi_{ij}^l \iff (e_i \otimes m_i)a = e_i \otimes (m_i a) + \sum_{j,l} r_{ij}^l \otimes \phi_{ij}^l(m_i)$$

Deformations in $\text{Def}_M(R)$ can therefore be represented by commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\eta_R} & (R_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \\ & \searrow \rho & \downarrow \\ & & \bigoplus_{i=1}^r \text{End}_k(M_i) \end{array}$$

These deformations are called *noncommutative deformations* since the base ring R is noncommutative.

The family $M = \{M_1, \dots, M_r\}$ is called a *swarm* if $\dim_k \text{Ext}_A^1(M, M)$ is finite dimensional over k . In this case, the noncommutative deformation functor Def_M has a pro-representing hull H and a versal family $M_H \in \text{Def}_M(H)$; see Laudal [2], Theorem 3.1. The defining property of the miniversal pro-couple (H, M_H) is that the induced natural transformation

$$\phi : \text{Mor}(H, -) \rightarrow \text{Def}_M$$

on \mathfrak{a}_r is smooth (which implies that ϕ_R is surjective for any R in \mathfrak{a}_r), and that ϕ_R is an isomorphism when $J(R)^2 = 0$. The miniversal pro-couple (H, M_H) is unique up to (non-canonical) isomorphism.

Let M be a swarm of right A -modules, and let (H, M_H) be the miniversal pro-couple of the noncommutative deformation functor $\text{Def}_M : \mathfrak{a}_r \rightarrow \text{Sets}$. We define the *algebra of observables* of M to be

$$\mathcal{O}(M) = \text{End}_H(M_H) \cong (H_{ij} \hat{\otimes}_k \text{Hom}_k(M_i, M_j))$$

and write $\eta : A \rightarrow \mathcal{O}(M)$ for induced *versal morphism*, giving the right A -module structure on M_H . By construction, it fits into the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & (H_{ij} \hat{\otimes}_k \text{Hom}_k(M_i, M_j)) \\ & \searrow \rho & \downarrow \\ & & \bigoplus_{i=1}^r \text{End}_k(M_i) \end{array}$$

Remark 1 Notice that the diagram extends the right action of A on the family \mathbf{M} to a right action of $\mathcal{O}(\mathbf{M})$, such that \mathbf{M} is a family of right $\mathcal{O}(\mathbf{M})$ -modules.

Remark 2 For any R in \mathfrak{a}_r and any deformation $M_R \in \text{Def}_{\mathbf{M}}(R)$, there is a morphism $u : H \rightarrow R$ in $\widehat{\mathfrak{a}}_r$ such that $\text{Def}_{\mathbf{M}}(u)(M_H) = M_R$ by the versal property, and the deformation M_R is therefore given by the composition $\eta_R = u^* \circ \eta$ in the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathcal{O}(\mathbf{M}) \\ & \searrow \eta_R & \downarrow u^* = u \otimes \text{id} \\ & & (R_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \end{array}$$

In this sense, the versal morphism $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$ determines all noncommutative deformations of the family \mathbf{M} .

3 Iterated extensions and matric Massey products

Let E be a right A -module and let $r \geq 1$ be a positive integer. If E has a *cofiltration* of length r , given by a sequence

$$E = E_r \xrightarrow{f_r} E_{r-1} \rightarrow \cdots \rightarrow E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 = 0$$

of surjective right A -module homomorphisms $f_i : E_i \rightarrow E_{i-1}$, then we call E an *iterated extension* of the right A -modules M_1, M_2, \dots, M_r , where $M_i = \ker(f_i)$. In fact, the cofiltration induces short exact sequences

$$0 \rightarrow M_i \rightarrow E_i \xrightarrow{f_i} E_{i-1} \rightarrow 0$$

for $1 \leq i \leq r$. Hence $E_1 \cong M_1$, E_2 is an extension of E_1 by M_2 , and in general, E_i is an extension of E_{i-1} by M_i . We write $\xi_{i-1,i} \in \text{Ext}_A^1(M_{i-1}, M_i)$ for the image of the extension above under the induced map $\text{Ext}_A^1(E_{i-1}, M_i) \rightarrow \text{Ext}_A^1(M_{i-1}, M_i)$.

We shall describe a necessary and sufficient condition for a cofiltration of the above type to exist, when the modules M_1, M_2, \dots, M_r and the extensions $\xi_{12}, \xi_{23}, \dots, \xi_{r-1,r}$ are given. The condition is given in terms of the *matric Massey products* of $\xi_{12}, \xi_{23}, \dots, \xi_{r-1,r}$ in the sense of May [3].

We consider the Hochschild complex $\text{HC}^\bullet(A, \text{End}_k(M))$ of A with values in $\text{End}_k(M)$ as a DGA (differential graded algebra) over k^r . It has decomposition

$$\text{HC}^n(A, \text{End}_k(M)) = \bigoplus_{i,j} \text{HC}^n(A, \text{Hom}_k(M_i, M_j))$$

A 1-cochain in this DGA is a k -linear map $\alpha : A \rightarrow \text{End}_k(M)$, and it is a 1-cocycle if and only if it is a derivation. Multiplication of the 1-cochains α, β in the DGA is defined by the composition $\alpha \cdot \beta = \{(a, b) \mapsto \beta(b) \circ \alpha(a)\}$. It is well-known that its cohomology is given by

$$\text{HH}^n(A, \text{End}_k(M)) = \bigoplus_{i,j} \text{HH}^n(A, \text{Hom}_k(M_i, M_j)) \cong \bigoplus_{i,j} \text{Ext}_A^n(M_i, M_j)$$

In particular, $\text{Ext}_A^1(M_i, M_j) \cong \text{Der}_k(A, \text{Hom}_k(M_i, M_j)) / \text{IDer}_k(A, \text{Hom}_k(M_i, M_j))$, where $\text{IDer}_k(-, -)$ denotes the inner derivations.

We choose a derivation $\alpha_{i-1,i} : A \rightarrow \text{Hom}_k(M_{i-1}, M_i)$ that represents $\xi_{i-1,i}$ in Hochschild cohomology for $2 \leq i \leq r$, and fix this choice. The *cup product* $\xi_{12} \cup \xi_{23} = \langle \xi_{12}, \xi_{23} \rangle$ is the cohomology class of $\alpha_{12} \cdot \alpha_{23}$. It is also called a second order matrix Massey product. If the cup-products $\langle \xi_{12}, \xi_{23} \rangle = \langle \xi_{23}, \xi_{34} \rangle = 0$, then there are 1-cochains α_{13} and α_{24} such that

$$d(\alpha_{13}) = \alpha_{12} \cdot \alpha_{23} \quad \text{and} \quad d(\alpha_{24}) = \alpha_{23} \cdot \alpha_{34}$$

In that case, $\underline{\alpha} = \{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{13}, \alpha_{24}\}$ is called a *defining system* for the third order *matrix Massey product* $\langle \xi_{12}, \xi_{23}, \xi_{34} \rangle$, and the cohomology class of

$$\tilde{\alpha}_{14} = \alpha_{13} \cdot \alpha_{34} + \alpha_{12} \cdot \alpha_{24}$$

is the corresponding value of $\langle \xi_{12}, \xi_{23}, \xi_{34} \rangle$. Notice that this cohomology class may depend on the defining system. Higher order matrix Massey products are defined similarly:

Definition 3 A defining system for the matrix Massey product $\langle \xi_{12}, \xi_{23}, \dots, \xi_{r-1,r} \rangle$ is a family

$$\underline{\alpha} = \{\alpha_{ij} : 1 \leq i < j \leq r, (i, j) \neq (1, r)\}$$

of 1-cochains $\alpha_{ij} : A \rightarrow \text{Hom}_k(M_i, M_j)$ such that $\alpha_{i-1,i}$ is a 1-cocycle that represents $\xi_{i-1,i}$ for $2 \leq i \leq r$, and such that

$$d(\alpha_{ij}) = \tilde{\alpha}_{ij}, \quad \text{with} \quad \tilde{\alpha}_{ij} = \sum_{l=i+1}^{j-1} \alpha_{il} \cdot \alpha_{lj}$$

when $j - i > 1$. The matrix Massey product $\langle \xi_{12}, \xi_{23}, \dots, \xi_{r-1,r} \rangle$ is defined if it has a defining system. In that case, $\langle \xi_{12}, \xi_{23}, \dots, \xi_{r-1,r} \rangle$ is the collection of cohomology classes represented by

$$\tilde{\alpha}_{1r} = \sum_{l=2}^{r-1} \alpha_{1l} \cdot \alpha_{lr}$$

for some defining system $\underline{\alpha}$.

Let E_2 be a right A -module that is an extension of M_1 by M_2 , such that there is a short exact sequence $0 \rightarrow M_2 \rightarrow E_2 \rightarrow M_1 \rightarrow 0$. Then it is well-known that $E_2 \cong M_2 \oplus M_1$ considered as a vector space over k , and that the right action of A is given by

$$(m_2, m_1)a = (m_2 \cdot a + \psi_a^{12}(m_1), m_1 \cdot a)$$

where $\psi^{12} : A \rightarrow \text{Hom}_k(M_1, M_2)$ is a k -linear map. Since the action of A must be associative, ψ^{12} must be a derivation. In fact, it is a derivation that represents the extension ξ_{12} .

Let E_3 be a right A -module that is an extension of E_2 by M_3 , such that there is a short exact sequence $0 \rightarrow M_3 \rightarrow E_3 \rightarrow E_2 \rightarrow 0$. Then $E_3 \cong M_3 \oplus E_2 \cong M_3 \oplus M_2 \oplus M_1$ considered as a vector space over k , and the right action of A is given by

$$(m_3, m_2, m_1)a = (m_3 \cdot a + \psi_a^{23}(m_2) + \psi_a^{13}(m_1), m_2 \cdot a + \psi_a^{12}(m_1), m_1 \cdot a)$$

where $\psi^{i3} : A \rightarrow \text{Hom}_k(M_i, M_3)$ is a k -linear map for $i = 1, 2$. Since the action of A must be associative, ψ^{23} must be a derivation (representing the extension ξ_{23}), and ψ^{13} must satisfy

$$-d(\psi^{13}) = \tilde{\psi}_{13}, \quad \text{with } \tilde{\psi}_{13} = \psi^{12} \cdot \psi^{23}$$

such that the cup product $\xi_{12} \cup \xi_{23} = 0$. It follows by an inductive argument that in the cofiltration

$$E = E_r \xrightarrow{f_r} E_{r-1} \rightarrow \cdots \rightarrow E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 = 0$$

we have that $E = E_r \cong M_r \oplus \cdots \oplus M_2 \oplus M_1$ considered as a vector space over k , with right action of A given by

$$(m_r, \dots, m_2, m_1)a = (m_r \cdot a + \sum_{i=1}^{r-1} \psi_a^{ir}(m_i), \dots, m_2 \cdot a + \psi_a^{12}(m_1), m_1 \cdot a)$$

where $\psi^{ij} : A \rightarrow \text{Hom}_k(M_i, M_j)$ is a 1-cochain for $1 \leq i < j \leq r$. The conditions that these cochains must satisfy for the action of A to be associative, is that

$$-d(\psi^{ij}) = \tilde{\psi}_{ij}, \quad \text{with } \tilde{\psi}_{ij} = \sum_{l=i+1}^{j-1} \psi^{il} \cdot \psi^{lj}$$

In other words, the family $\underline{\alpha} = \{\alpha_{ij} : 1 \leq i < j \leq r, (i, j) \neq (1, r)\}$ given by $\alpha_{ij} = (-1)^{j-i+1} \psi^{ij}$ is a defining system for the matrix Massey product

$$\langle \xi_{12}, \xi_{23}, \dots, \xi_{r-1, r} \rangle$$

Moreover, the cohomology class of $\tilde{\alpha}_{1r}$ is zero, since $\tilde{\alpha}_{1r} = d(\alpha_{1r})$. This proves the following result:

Proposition 4 *Let M_1, \dots, M_r be right A -modules, and let $\xi_{i-1, i} \in \text{Ext}_A^1(M_{i-1}, M_i)$ for $2 \leq i \leq r$. There is an iterated extension E of M_1, \dots, M_r with induced extensions $\xi_{12}, \xi_{23}, \dots, \xi_{r-1, r}$ if and only if the matrix Massey product*

$$\langle \xi_{12}, \xi_{23}, \dots, \xi_{r-1, r} \rangle$$

is defined and contains zero.

Corollary 5 *Let E be an iterated extension of M_1, \dots, M_r with induced extensions $\xi_{12}, \dots, \xi_{r-1, r}$, and write $\{\alpha_{ij} : 1 \leq i < j \leq r\}$ for the family of cochains consisting of a defining system for the matrix Massey product $\langle \xi_{12}, \xi_{23}, \dots, \xi_{r-1, r} \rangle$ and the cochain α_{1r} such that $d(\alpha_{1r}) = \tilde{\alpha}_{1r}$ represents the product. If $K \subseteq A$ is an ideal that satisfies the conditions*

1. $M_i \cdot K = 0$ for $1 \leq i \leq r$
2. $\alpha_{ij}(K) = 0$ for $1 \leq i < j \leq r$

then $E \cdot K = 0$.

4 Injectivity

Let $\mathbf{M} = \{M_1, \dots, M_r\}$ be a swarm of right A -modules, and let $\text{Def}_{\mathbf{M}} : \mathbf{a}_r \rightarrow \text{Sets}$ be its noncommutative deformation functor. Then $\text{Def}_{\mathbf{M}}$ has a miniversal pro-couple (H, M_H) , and we consider the induced algebra homomorphism $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$ and its kernel $K = \ker(\eta)$.

Lemma 6 *Let \mathbf{M} be a swarm of right A -modules. For any iterated extension E of the family \mathbf{M} , we have that $E \cdot K = 0$, where $K = \ker(\eta)$ is the kernel of $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$.*

We claim that K satisfies the conditions in Corollary 5, and this is enough to prove Lemma 6. We shall prove the claim later in this section. However, let us first show that Lemma 6 implies the injectivity of η in the Generalized Burnside Theorem:

Corollary 7 *If A , considered as a right A -module, is an iterated extension of a swarm \mathbf{M} , then the versal morphism $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$ is injective. In particular, η is injective when A is a finite dimensional algebra and \mathbf{M} is the family of simple right A -modules.*

Proof If A is an iterated extension of \mathbf{M} , then $1 \cdot K = 0$ by Lemma 6, and this implies that $K = 0$. If A is finite dimensional, then the right A -module A has finite length, and it is an iterated extension of the simple modules. \square

With this done, we return to the claim that $K = \ker(\eta)$ satisfies the conditions in Corollary 5 for any iterated extension E of \mathbf{M} . We shall now prove the claim, by constructing (H, ξ) explicitly using generalized matrix Massey products. This will complete the proof of Lemma 6, and therefore complete the proof of the injectivity of η in the Generalized Burnside Theorem.

Let us write (H_n, ξ_n) for the couple consisting of the algebra $H_n = H/J(H)^n$ and the versal family $\xi_n \in \text{Def}_{\mathbf{M}}(H_n)$. We have that $K = \cap_n K_n$, where K_n is the kernel of the algebra homomorphism

$$\eta_n : A \rightarrow ((H_n)_{ij} \otimes_k \text{Hom}_k(M_i, M_j))$$

corresponding to $\xi_n \in \text{Def}_{\mathbf{M}}(H_n)$. In particular, $H_1 \cong k^r$, and we may identify η_1 with ρ . This implies that $K_1 = \ker(\rho) = \{a \in A : M_1 \cdot a = \dots = M_r \cdot a = 0\}$, and proves that K satisfies the first condition in Corollary 5.

In order to describe K_n for $n \geq 2$ and show that K satisfies the second condition in Corollary 5, we need an explicit description of (H_n, ξ_n) . We may, without loss of generality, assume that $\text{Ext}_A^n(M_i, M_j)$ is a finite dimensional vector space for $1 \leq i, j \leq r$, $n = 1, 2$. We let $V_{ij}^n = \text{Ext}_A^n(M_i, M_j)^*$ be its dual vector space, and consider $V^n = (V_{ij}^n)$ as a k^r -bimodule. Its tensor algebra $\mathbb{T}(V^n)$ over k^r , explicitly given by

$$\mathbb{T}(V^n) = \bigoplus_{m \geq 0} \mathbb{T}^m(V^n) = k^r \oplus V^n \oplus (V^n \otimes_{k^r} V^n) \oplus \dots \oplus (\otimes_{k^r}^m V^n) \oplus \dots$$

is as an r -pointed algebra, with $\mathbb{T}(V^n) \rightarrow k^r$ given by $V^n \mapsto 0$. We define \mathbb{T}^n to be the completion of $\mathbb{T}(V^n)$ in the pro-category $\widehat{\mathbf{a}}_r$ for $n = 1, 2$.

Example 8 Assume that $r = 2$ and that $d_{ij} = \dim_k V_{ij}^1 = \dim_k \text{Ext}_A^1(M_i, M_j)^*$ is given by $d_{11} = d_{12} = 2$, $d_{21} = 0$ and $d_{22} = 1$. Then the matrix algebras defined above are given by

$$\mathsf{T}(V^1) = \begin{pmatrix} k\langle x, y \rangle & (u, v) \\ 0 & k[z] \end{pmatrix}, \quad \mathsf{T}^1 = \begin{pmatrix} k\langle\langle x, y \rangle\rangle & (u, v) \\ 0 & k[[z]] \end{pmatrix}$$

where $\{x, y\} \subseteq \text{Ext}_A^1(M_1, M_1)^*$, $\{u, v\} \subseteq \text{Ext}_A^1(M_1, M_2)^*$ and $\{z\} \subseteq \text{Ext}_A^1(M_2, M_2)^*$ are k -linear bases, and where (u, v) denotes the free bimodule generated by $\{u, v\}$.

At the tangent level, we have that $H_2 = \mathsf{T}^1/J(\mathsf{T}^1)^2$ and that $\xi_2 \in \text{Def}_M(H_2)$ is the deformation given by the right action

$$(e_i \otimes m_i)a = e_i \otimes (m_i a) + \sum_{j,l} t_{ij}(l) \otimes \psi_{ij}^l(a)(m_i)$$

where $\{t_{ij}(l) : l\}$ is a base of V_{ij}^1 and $\psi_{ij}^l : A \rightarrow \text{Hom}_k(M_i, M_j)$ is a derivation that represents $t_{ij}(l)^*$ in Hochschild cohomology. In other words, ξ_2 is the deformation corresponding to η_2 , given by

$$a \mapsto \sum_i e_i \otimes \rho_i(a) + \sum_{i,j,l} t_{ij}(l) \otimes \psi_{ij}^l(a)$$

It follows that $K_2 = \ker(\eta_2)$ consists of all $a \in K_1$ such that $\psi(a) = 0$ for all derivations $\psi : A \rightarrow \text{End}_k(M)$. Given an iterated extension E of M and its induced extensions $\xi_{i-1,i}$ for $2 \leq i \leq n$, it follows from Proposition 4 that the matrix Massey product $\langle \xi_{12}, \xi_{23}, \dots, \xi_{r-1,r} \rangle$ is defined and contains zero, and we fix a defining system

$$\underline{\alpha} = \{\alpha_{ij} : 1 \leq i < j \leq r, (i, j) \neq (1, r)\}$$

together with a cochain α_{1r} such that $\tilde{\alpha}_{1r} = d(\alpha_{1r})$. From the description of K_2 given above, it follows that $\alpha_{i-1,i}(K) = 0$ for $2 \leq i \leq n$.

It remains to prove that $\alpha_{i,j}(K) = 0$ also when $1 \leq i < j \leq n$ with $j - i \geq 2$. For this, we need an explicit description of (H_n, ξ_n) for $n \geq 3$, given below in terms of generalized matrix Massey products. For further details on the general method, we refer to Laudal [2], Eriksen [1], and Siqveland [4].

There is an obstruction morphism $o : \mathsf{T}^2 \rightarrow \mathsf{T}^1$ in $\widehat{\mathbf{a}}_r$ such that $H \cong \mathsf{T}^1/a$ with $a = o(J(\mathsf{T}^2))$; see Theorem 3.1 in Laudal [2] or Proposition 5.1 in Eriksen [1]. The idea is that $a \subseteq \mathsf{T}^1$ is the minimal ideal of obstructions for lifting $\xi_2 \in \text{Def}_M(H_2)$ to T^1 via the natural morphism $\mathsf{T}^1 \rightarrow H_2$. If we choose a k -base $\{s_{ij}(l) : l\}$ of V_{ij}^2 and let $f_{ij}(l) = o(s_{ij}(l))$, then $a = (f_{ij}(l))$. For $n \geq 2$, we write $f_{ij}^n(l)$ for the truncated noncommutative power series obtained as the images of $f_{ij}(l)$ in $\mathsf{T}^1/J(\mathsf{T}^1)^{n+1}$. These power series are not unique, but their leading terms are.

Let M^* be the restriction of $o : \mathsf{T}^2 \rightarrow \mathsf{T}^1$ to $V^2 \subseteq \mathsf{T}^2$. Moreover, for $n \geq 2$, let M^n be the projection of M^* on the first $n - 1$ factors, and let M_n be the dual map of M^n . We obtain homomorphisms of k^r -bimodules

$$M^n : V^2 \rightarrow \prod_{i=2}^n \mathsf{T}^i(V^1), \quad M_n : \prod_{i=2}^n \mathsf{T}^i(\text{Ext}_A^1(M, M)) \rightarrow \text{Ext}_A^2(M, M)$$

The map M^n is given by $s_{ij}(l) \mapsto f_{ij}^n(l)$ and is not unique, but its image modulo $\Delta_n = V^1 \cdot \text{im}(M^{n-1}) + \text{im}(M^{n-1}) \cdot V^1 \subseteq T^1$ is unique. Therefore, the quotient map

$$M_o^n : V^2 \rightarrow \prod_{i=2}^n T^i(V^1)/\Delta_n$$

is uniquely defined, and we let $M_n^o : D_n \rightarrow \text{Ext}_A^2(M, M)$ be its dual map, with

$$D_n = \left(\prod_{i=2}^n T^i(V^1)/\Delta_n \right)^* \subseteq \prod_{i=2}^n T^i(\text{Ext}_A^1(M, M))$$

The homomorphisms $M_n^o : D_n \rightarrow \text{Ext}_A^2(M, M)$ of k^r -bimodules are the *generalized matrix Massey products* induced by the obstruction morphism o . We shall explain how M_n^o can be expressed in terms of the matrix Massey products of Section 3.

Clearly, we have that $D_2 = \text{Ext}_A^1(M, M) \otimes \text{Ext}_A^1(M, M)$, and that $M_2^o = M_2$. Moreover, M_2 is defined by the obstructions for lifting ξ_2 to $T_3^1 = T^1/J(T^1)^3$. We recall that ξ_2 is given by the right action

$$(e_i \otimes m_i)a = e_i \otimes (m_i a) + \sum_{j,l} t_{ij}(l) \otimes \psi_{ij}^l(a)(m_i) \quad (1)$$

and the associativity of this action in T_3^1 is the obstruction for lifting ξ_2 . Since

$$((e_i \otimes m_i)a)b - (e_i \otimes m_i)(ab) = \sum_{k,l'} \sum_{j,l} t_{ij}(l) \cdot t_{jk}(l') \otimes \left(\psi_{jk}^{l'}(b) \circ \psi_{ij}^l(a) \right) (m_i)$$

we find that $M_2^o(\psi_{ij}^l, \psi_{jk}^{l'}) = \langle \psi_{ij}^l, \psi_{jk}^{l'} \rangle$. The obstructions in T_3^1 are then given by

$$f_{ik}^2(l'') = \sum_{j,l,l'} s_{ik}(l'') \left(M_2^o(\psi_{ij}^l, \psi_{jk}^{l'}) \right) \cdot t_{ij}(l) t_{jk}(l')$$

and $H_3 = T^1/a_3$, where $a_3 = J(T^1)^3 + (f_{ik}^2(l''))$. We choose a k -linear base of elements in $\ker(M_2^o)_{ik}$ of the form

$$\sum_{j,l,l'} c_{j,l,l'} t_{ij}(l)^* \otimes t_{jk}(l')^*$$

For each of these base elements, there is a 1-cochain $\psi_{ik} \in \text{Hom}_k(A, \text{Hom}_k(M_i, M_k))$ such that

$$-d(\psi_{ik})(a, b) = \sum_{j,l,l'} c_{j,l,l'} \psi_{jk}(l')(b) \circ \psi_{ij}(l)(a)$$

To lift ξ_2 to a deformation $\xi_3 \in \text{Def}_M(H_3)$, we have to adjust the expression in Equation 1 by adding a term of the form

$$\left(\sum_{j,l,l'} c_{j,l,l'} t_{ij}(l) t_{jk}(l') \right) \otimes \psi_{ik}(a)(m_i)$$

for each base element of $\ker(M_2^o)_{ik}$. It follows that $K_3 = \ker(\eta_3)$ consists of all $a \in K_2$ such that $\psi_{ij}(a) = 0$ for all such ψ_{ik} corresponding to base elements in $\ker(M_2^o)_{ik}$. In particular, $\alpha_{ij}(K) = 0$ whenever $j - i = 2$.

We have to continue in this way, lifting ξ_n to ξ_{n+1} for $n \geq 3$. At each step, the associativity condition is the obstruction for lifting, and it can be expressed in terms of the matric Massey products of Section 3, and $K_{n+1} = \ker(\eta_{n+1})$ consists of all $a \in K_n$ such that $\psi(a) = 0$ for all cochains ψ corresponding to base elements in $\ker(M_n^o)$. Since $M_n^o(\alpha_{i,i+1} \otimes \cdots \otimes \alpha_{j-1,j}) = 0$, it follows that $\alpha_{ij}(K) = 0$ whenever $j - i = n$. Hence, K satisfies both conditions of Corollary 5.

Example 9 Let A be a quotient of the algebra $\mathbb{T}(V^1)$ from Example 8 by the ideal (f, g) , where $f = yu - xv + 2vz + uz^2$ and $g = xy - yx$, given by

$$A = \mathbb{T}(V^1)/(f, g) \cong \begin{pmatrix} k[x, y] & (u, v) \\ 0 & k[z] \end{pmatrix} / (f)$$

We consider the family $\mathbf{M} = \{M_1, M_2\}$ of right A -modules, where M_i is a right A -module via A_{ii} , given by $M_1 = k[x, y]/(x, y)$ and $M_2 = k[z]/(z)$. Then \mathbf{M} is a family of simple one-dimensional right A -modules, and we shall indicate how the miniversal couple (H, M_H) of $\text{Def}_{\mathbf{M}}$ can be constructed. Recall that

$$\text{Ext}_A^1(M_i, M_j) \cong \text{Der}_k(A, \text{Hom}_k(M_i, M_j)) / \text{IDer}_k(A, \text{Hom}_k(M_i, M_j))$$

We may identify $\text{Hom}_k(M_i, M_j) \cong k$, and any derivation of A is determined by its values on the generators of A . We find that

$$\begin{aligned} \text{Ext}_A^1(M_1, M_1) &= k\langle \psi_{11}^1, \psi_{11}^2 \rangle \cong k^2 & \text{Ext}_A^1(M_1, M_2) &= k\langle \psi_{12}^1, \psi_{12}^2 \rangle \cong k^2 \\ \text{Ext}_A^1(M_2, M_1) &= 0 & \text{Ext}_A^1(M_2, M_2) &= k\langle \psi_{22} \rangle \cong k \end{aligned}$$

where $\psi_{11}^1, \psi_{11}^2, \psi_{12}^1, \psi_{12}^2, \psi_{22}$ are the derivations corresponding to the generators x, y, u, v, z , such that for instance $\psi_{11}^1(a) = 1$ if $a = x$, and $\psi_{11}^1(a) = 0$ for the other generators y, u, v, z . We choose dual bases $\{t_{ij}(l)\}$ for $\text{Ext}_A^1(M_i, M_j)^*$, where we write $t_{22} = t_{22}(1)$ for simplicity, and define $\alpha(t_{ij}(l)) = \psi_{ij}^l$. Then the family $\underline{\alpha}$ is a defining system for the second order matric Massey products:

$$\begin{aligned} \langle \psi_{11}^1, \psi_{11}^1 \rangle &= -d((x^2)^*) & \langle \psi_{11}^1, \psi_{12}^1 \rangle &= -d((xu)^*) \\ \langle \psi_{11}^1, \psi_{11}^2 \rangle &= s_{11}^* & \langle \psi_{11}^1, \psi_{12}^2 \rangle &= s_{12}^* \\ \langle \psi_{11}^2, \psi_{11}^1 \rangle &= -s_{11}^* - d((xy)^*) & \langle \psi_{11}^2, \psi_{12}^1 \rangle &= -s_{12}^* - d((yu)^*) \\ \langle \psi_{11}^2, \psi_{11}^2 \rangle &= -d((y^2)^*) & \langle \psi_{11}^2, \psi_{12}^2 \rangle &= -d((yv)^*) \\ \\ \langle \psi_{12}^1, \psi_{22} \rangle &= -d((uz)^*) & \langle \psi_{22}, \psi_{22} \rangle &= -d((z^2)^*) \\ \langle \psi_{12}^2, \psi_{22} \rangle &= -2s_{12}^* - d((vz)^*) \end{aligned}$$

We let the 2-cocycles s_{11}^*, s_{12}^* be defined by the matric Massey products above, determining a linear subspace of $\text{Ext}_A^2(M, M)$ of dimension two. The obstructions at the next level are then given by

$$\begin{aligned} f_{11}^2 &= t_{11}(1) \cdot t_{11}(2) - t_{11}(2) \cdot t_{11}(1) \\ f_{12}^2 &= t_{11}(1) \cdot t_{12}(2) - t_{11}(2) \cdot t_{12}(1) - 2t_{12}(2) \cdot t_{22} \end{aligned}$$

such that $H_3 = \mathbb{T}_3^1/(f_{11}^2, f_{12}^2)$. We may choose a base for $\ker(M_2^o)$ and a cochain $\alpha(t)$ for each base element t , such that the ‘‘adjusted’’ actions obtained by adding

the $\alpha(t)$'s define a lifting of ξ_2 to $\xi_3 \in \text{Def}_M(H_3)$. For instance, we may choose $\alpha(t_{11}(1)^2) = (x^2)^*$. At the next level, the only third order matrix Massey product that is non-zero, is given by

$$\langle \psi_{12}^1, \psi_{22}, \psi_{22} \rangle = -s_{12}^*$$

This implies that $f_{11}^3 = f_{11}^2$ and $f_{12}^3 = f_{12}^2 - t_{12}(1)t_{22}^2$. Continuing in this way, we find that

$$H = \begin{pmatrix} k[[t_{11}(1), t_{11}(2)]] & (t_{12}(1), t_{12}(2)) \\ 0 & k[[t_{22}]] \end{pmatrix} / (f_{12})$$

where $f_{12} = t_{11}(1)t_{12}(2) - t_{11}(2)t_{12}(1) - 2t_{12}(2)t_{22} - t_{12}(1)t_{22}^2$. This illustrates how to construct (H, M_H) in the general case. In particular, it shows how $K = \ker(\eta)$ is determined by M_n^o , and therefore by the matrix Massey products.

5 The Generalized Burnside Theorem

Let A be a finite dimensional k -algebra, and let $M = \{M_1, \dots, M_r\}$ be the family of simple right A -modules (up to isomorphism). We consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \\ & \searrow \rho & \downarrow \\ & & \bigoplus_{i=1}^r \text{End}_k(M_i) \end{array}$$

Clearly, ρ factors through $A/J(A)$, and $A/J(A) \rightarrow \bigoplus_i \text{End}_k(M_i)$ is an isomorphism when $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$ by the Artin-Wedderburn theory for semisimple algebras. This proves the version of the Classical Burnside Theorem mentioned in the introduction.

In this section, we shall prove the Generalized Burnside Theorem. Notice that $\eta : A \rightarrow \mathcal{O}(M)$ maps the Jacobson radical $J(A)$ of A to the Jacobson radical $J = (J(H)_{ij} \otimes_k \text{Hom}_k(M_i, M_j))$ of $\mathcal{O}(M)$. Moreover, A is complete in the $J(A)$ -adic topology since it is finite dimensional, and $\mathcal{O}(M)$ is clearly J -adic complete. The Classical Burnside Theorem can therefore be interpreted as the statement that the first order approximation $\text{gr}_0(\eta) : A/J(A) \rightarrow \mathcal{O}(M)/J$ of η is an isomorphism.

Theorem 10 (Generalized Burnside Theorem) *Let A be a finite dimensional algebra and let M be the family of simple right A -modules, up to isomorphism. Then the versal morphism $\eta : A \rightarrow \mathcal{O}(M)$ is injective, and if $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then η is an isomorphism. In particular, η is an isomorphism if k is algebraically closed.*

Proof It follows from Corollary 7 that η is injective. To prove that η is surjective, it is enough to show that $\hat{\eta} : \hat{A} \rightarrow \hat{\mathcal{O}}(M)$ is surjective, since the comments above show that A and $\mathcal{O}(M)$ are complete. By a standard result for filtered algebras, it is sufficient to show that $\text{gr}_1(\eta) : J(A)/J(A)^2 \rightarrow J/J^2$ is surjective, since $\text{gr}_0(\eta)$ is an isomorphism by the Classical Burnside Theorem. We notice that

$$J/J^2 \cong ((J(H)/J(H)^2)_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) \cong (\text{Ext}_A^1(M_i, M_j)^* \otimes_k \text{Hom}_k(M_i, M_j))$$

since $J(H)/J(H)^2$ is the dual of the tangent space $(\text{Ext}_A^1(M_i, M_j))$ of Def_M . By the Classical Burnside Theorem, we have that

$$A/J(A) \cong \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i)$$

and therefore $A/J(A)$ is separable. By Wedderburn's Principal Theorem, it therefore follows that there is a semisimple subalgebra $S \subseteq A$ such that $A = J(A) \oplus S$. In particular, any $a \in A$ can be written as $a = j + s$ with $j \in J(A)$, $s \in S$, and this decomposition is unique. We claim that

$$\text{Ext}_A^1(M_i, M_j) \cong \text{Hom}_{A-A}(J(A)/J(A)^2, \text{Hom}_k(M_i, M_j))$$

for all i, j . Recall that $\text{Ext}_A^1(M_i, M_j) \cong \text{Der}_k(A, \text{Hom}_k(M_i, M_j))/\text{im}(d^0)$, where $\text{im}(d^0) = \text{IDer}_k(A, \text{Hom}_k(M_i, M_j))$ are the inner derivations. We shall use this to prove the claim above. Define a map

$$u : \text{Der}_k(A, \text{Hom}_k(M_i, M_j)) \rightarrow \text{Hom}_{A-A}(J(A)/J(A)^2, \text{Hom}_k(M_i, M_j))$$

where $u(D)(\bar{x}) = D(x)$ for all $x \in J(A)$. Since $D(J(A)^2) = 0$, we see that $u(D)$ is well-defined, and $u(D)$ is clearly A - A bilinear. We shall prove that $\text{im}(d^0) = \ker(u)$: Any inner derivation $D = d^0(\phi)$ clearly gives $u(D) = 0$. Conversely, if $u(D) = 0$, then there is an induced derivation $\bar{D} : A/J(A) \rightarrow \text{Hom}_k(M_i, M_j)$, and $\bar{D} = d^0(\phi)$ is inner since $A/J(A)$ is semisimple. This means that D is also inner. Finally, we show that u is surjective: For any A - A bilinear map $\phi : J(A)/J(A)^2 \rightarrow \text{Hom}_k(M_i, M_j)$, we define $D(a) = \phi(\bar{j})$, where $a = j + s$ is the decomposition mentioned above. If $b = j' + s'$ is the decomposition of another element $b \in A$, then

$$ab = (j + s)(j' + s') = jj' + js' + sj' + ss'$$

is the decomposition of ab , with $jj' + js' + sj' \in J(A)$, $ss' \in S$. Therefore, it follows that $D : A \rightarrow \text{Hom}_k(M_i, M_j)$ is a derivation. In fact, we have that

$$\begin{aligned} aD(b) + D(a)b &= a\phi(\bar{j'}) + \phi(\bar{j})b = (j + s)\phi(\bar{j'}) + \phi(\bar{j})(j' + s') \\ &= \phi(\overline{sj' + js'}) = D(ab) \end{aligned}$$

It is clear that $u(D) = \phi$, and therefore u is surjective. When we write $\bar{A} = A/J(A)$, this implies that we have

$$\begin{aligned} \text{Ext}_A^1(M_i, M_j) &\cong \text{Hom}_{A-A}(J(A)/J(A)^2, \text{Hom}_k(M_i, M_j)) \\ &\cong \text{Hom}_{\bar{A}-\bar{A}}(J(A)/J(A)^2, \text{Hom}_k(M_i, M_j)) \end{aligned}$$

Since $S \cong A/J(A) \cong \bigoplus_i \text{End}_k(M_i)$, there are idempotents e_1, \dots, e_r in $S \subseteq A$ corresponding to the identities on M_1, M_2, \dots, M_r , and we have that $e_i e_j = 0$ for $i \neq j$ and $1 = e_1 + \dots + e_r$. Let $A = (A_{ij})$ be the corresponding decomposition of A , with $A_{ii}/J(A)_{ii} \cong \text{End}_k(M_i)$. Let $E = J(A)/J(A)^2$ and $E_{ij} = e_i E e_j$. Then $J(A)/J(A)^2 = (E_{ij})$, where E_{ij} is an $\text{End}_k(M_i)$ - $\text{End}_k(M_j)$ bimodule, and therefore there is a vector space W_{ij} of finite dimension over k such that

$$E_{ij} \cong M_i^* \otimes W_{ij} \otimes M_j \cong W_{ij} \otimes_k \text{Hom}_k(M_i, M_j)$$

Let $n_{ij} = \dim_k W_{ij}$. Then we have isomorphisms

$$\begin{aligned}
\text{Ext}_A^1(M_i, M_j) &\cong \text{Hom}_{\overline{A}-\overline{A}}(J(A)/J(A)^2, \text{Hom}_k(M_i, M_j)) \\
&\cong \text{Hom}_{\overline{A}-\overline{A}}(E_{ij}, \text{Hom}_k(M_i, M_j)) \\
&\cong \text{Hom}_{\text{End}_k(M_i)-\text{End}_k(M_j)}(W_{ij} \otimes_k \text{Hom}_k(M_i, M_j), \text{Hom}_k(M_i, M_j)) \\
&\cong W_{ij}^* \otimes_k \text{End}_{\text{End}_k(M_i)-\text{End}_k(M_j)}(\text{Hom}_k(M_i, M_j)) \\
&\cong W_{ij}^* \otimes_k k \cong W_{ij}^*
\end{aligned}$$

This implies that

$$J(A)/J(A)^2 = (E_{ij}) \cong (\text{Ext}_A^1(M_i, M_j)^* \otimes_k \text{Hom}_k(M_i, M_j)) \cong J/J^2$$

It follows that $\text{gr}_1(\eta) : J(A)/J(A)^2 \rightarrow J/J^2$ is an isomorphism. \square

6 The closure property

Let $A = k\langle x_1, \dots, x_d \rangle / I$ be a finitely generated k -algebra, and let $\mathbf{M} = \{M_1, \dots, M_r\}$ be a family of finite dimensional right A -modules. Then \mathbf{M} is a swarm, since

$$\dim_k \text{Ext}_A^1(M_i, M_j) \leq \dim_k \text{Der}_k(A, \text{Hom}_k(M_i, M_j)) \leq \dim_k \text{Hom}_k(M_i, M_j)^d$$

is finite. The fact that any derivation $D : A \rightarrow \text{Hom}_k(M_i, M_j)$ is determined by $D(x_l) \in \text{Hom}_k(M_i, M_j)$ for $1 \leq l \leq d$ gives the last inequality. We may therefore consider the algebra of observables $B = \mathcal{O}(\mathbf{M})$ of the swarm \mathbf{M} of right A -modules, and write $\eta : A \rightarrow B$ for the versal morphism. In general, $\mathbf{M} = \{M_1, \dots, M_r\}$ is a family of right B -modules via η .

Lemma 11 *The family $\mathbf{M} = \{M_1, \dots, M_r\}$ of right B -modules is the simple right B -modules, and it is swarm of B -modules.*

Proof It follows from the Artin-Wedderburn theory that $\mathbf{M} = \{M_1, \dots, M_r\}$ is the family of simple modules over

$$\overline{B} = B/J(B) \cong (H/J(H) \otimes_k \text{Hom}_k(M_i, M_j)) \cong \bigoplus_i \text{End}_k(M_i)$$

Since B and $\overline{B} = B/J(B)$ have the same simple modules, it follows that \mathbf{M} is the family of simple right B -modules. We have that $\text{Ext}_B^1(M_i, M_j)$ is a quotient of $\text{Der}_k(B, \text{Hom}_k(M_i, M_j))$, and any derivation $D : B \rightarrow \text{Hom}_k(M_i, M_j)$ satisfies $D(J^2) = JD(J) + D(J)J = 0$ when $J = J(B)$ since \mathbf{M} is the family of simple B -modules. From the fact that

$$B/J^2 \cong ((H/J(H)^2)_{ij} \otimes_k \text{Hom}_k(M_i, M_j))$$

is finite dimensional, and in particular a finitely generated k -algebra, it follows from the argument preceding the lemma that \mathbf{M} is a swarm of B -modules. \square

In this situation, we may iterate the process. Since \mathbf{M} is a swarm of right B -modules, the noncommutative deformation functor $\text{Def}_{\mathbf{M}}^B$ of \mathbf{M} , considered as a family of right B -modules, has a miniversal pro-couple (H^B, M_H^B) . We write $\mathcal{O}^B(\mathbf{M}) = \text{End}_{H^B}(M_H^B) \cong (H_{ij}^B \otimes_k \text{Hom}_k(M_i, M_j))$ for its algebra of observables and $\eta^B : B \rightarrow \mathcal{O}^B(\mathbf{M})$ for its versal morphism.

Theorem 12 *Let A be a finitely generated k -algebra, and let \mathbf{M} be a family of finite dimensional right A -modules. If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $\eta^B : B \rightarrow \mathcal{O}^B(\mathbf{M})$ is an isomorphism for $B = \mathcal{O}(\mathbf{M})$.*

Proof Since \mathbf{M} is a swarm of A -modules and of B -modules, we may consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\eta} & B = \mathcal{O}(\mathbf{M}) & \xrightarrow{\eta^B} & C = \mathcal{O}^B(\mathbf{M}) \\ & \searrow \rho & \downarrow & \swarrow & \\ & & \bigoplus_i \text{End}_k(M_i) & & \end{array}$$

The algebra homomorphism η^B induces maps $B/J(B)^n \rightarrow C/J(C)^n$ for all $n \geq 1$, and it is enough to show that each of these induced maps is an isomorphism. For $n = 1$, we have

$$B/J(B) \cong C/J(C) \cong \bigoplus_i \text{End}_k(M_i)$$

so it is clearly an isomorphism for $n = 1$. For $n \geq 2$, we have that $B_n = B/J(B)^n$ is a finite dimensional algebra with the same simple modules as B since $M_i J^n = 0$. We may therefore consider the versal morphism of the swarm \mathbf{M} of right B_n -modules, which is an isomorphism by the Generalized Burnside Theorem since $\text{End}_{B_n}(M_i) = k$ for $1 \leq i \leq r$. Finally, any derivation $D : B \rightarrow \text{Hom}_k(M_i, M_j)$ satisfies $D(J^n) = 0$ when $n \geq 2$. Therefore, we have that

$$\text{Ext}_{B_n}^1(M_i, M_j) \cong \text{Ext}_B^1(M_i, M_j)$$

This implies that $B/J(B)^n \rightarrow C/J(C)^n$ coincides with the versal morphism of the swarm \mathbf{M} of right B_n -modules, and therefore it is an isomorphism. \square

Theorem 12 implies that the assignment $(A, \mathbf{M}) \mapsto (B, \mathbf{M})$ is a closure operation when A is a finitely generated k -algebra and $\mathbf{M} = \{M_1, \dots, M_r\}$ is a family of finite dimensional right A -modules such that $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$. In other words, the algebra $B = \mathcal{O}(\mathbf{M})$ has the following properties:

1. The family \mathbf{M} is the family of simple right B -modules.
2. The family \mathbf{M} has exactly the same module-theoretic properties, in terms of extensions and matric Massey products, considered as a family of modules over B as over A .

Moreover, these properties characterize the algebra of observables $B = \mathcal{O}(\mathbf{M})$.

Remark 13 Assume that k is a field that is not algebraically closed. When A is a finite dimensional k -algebra and \mathbf{M} is the family of simple right A -modules, it could happen that the division algebra $D_i = \text{End}_A(M_i)$ has dimension $\dim_k D_i > 1$ for some simple A -modules M_i . In this case, $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$ is not necessarily an isomorphism. If we consider the subfamily $\mathbf{M}' = \{M_i : \text{End}_A(M_i) = k\} \subseteq \mathbf{M}$ and the algebra $B = \mathcal{O}(\mathbf{M}')$, this would be the “correct” algebra for the family \mathbf{M}' , and by the closure property, the Generalized Burnside Theorem holds for the family \mathbf{M}' of right B -modules.

7 Applications

Let A be a finite dimensional k -algebra. We consider the family $\mathbf{M} = \{M_1, \dots, M_r\}$ of simple right A -modules. By the Generalized Burnside Theorem, A can be written in *standard form* as

$$A \cong \text{im}(\eta) \subseteq (H_{ij} \otimes_k \text{Hom}_k(M_i, M_j)) = \mathcal{O}(\mathbf{M})$$

If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then the standard form of A is $A \cong \mathcal{O}(\mathbf{M})$, and in general, it is a subalgebra of $\mathcal{O}(\mathbf{M})$.

The standard form can, for instance, be used to compare finite dimensional algebras and determine if they are isomorphic. Let us illustrate this with a simple example. Let k be a field, and let $A = k[G]$ be the group algebra of $G = \mathbb{Z}_3$. In concrete terms, we have that $A \cong k[x]/(x^3 - 1)$, and over the algebraic closure of k , we have that

$$x^3 - 1 = (x - 1)(x^2 + x + 1) = (x - 1)(x - \omega)(x - \omega^2)$$

If $\text{char}(k) \neq 3$ and $\omega \in k$, then the simple A -modules are $\mathbf{M} = \{M_0, M_1, M_2\}$, where $M_i = A/(x - \omega^i)$. Furthermore, a calculation shows that

$$\text{Ext}_A^1(M_i, M_j) = 0 \text{ for } 0 \leq i, j \leq 2$$

Hence, the noncommutative deformation functor $\text{Def}_{\mathbf{M}}$ has a pro-representing hull $H = k^3$ (it is rigid), and that the versal morphism $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$ is an isomorphism. The standard form of A is therefore given by

$$A = k[\mathbb{Z}_3] \cong k^3 = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

If $\text{char}(k) = 3$, then M_0 is the only simple A -module since $x^3 - 1 = (x - 1)^3$, and we find that $\text{Ext}_A^1(M_0, M_0) = k$. In this case, it turns out that $H \cong k[[t]]/(t^3)$, and the standard form of A is given by $A = k[\mathbb{Z}_3] \cong k[t]/(t^3)$. In both cases, it follows from the Generalized Burnside Theorem that η is an isomorphism, since $\text{End}_A(M) = k$ for all the simple A -modules M .

If $\text{char}(k) \neq 3$ and $\omega \notin k$, then the simple A -modules are $\mathbf{M} = \{M, N\}$, where $M = M_0 = A/(x - 1)$ is 1-dimensional, and $N = A/(x^2 + x + 1) \cong k(\omega) = K$ is 2-dimensional. In this case, we have that $\text{End}_A(M) = k$ and $\text{End}_A(N) = K$, and we find that the standard form of A is given by

$$H = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \Rightarrow A \cong \text{im}(\eta) = \begin{pmatrix} k & 0 \\ 0 & K \end{pmatrix} \subseteq \mathcal{O}(\mathbf{M}) = \begin{pmatrix} k & 0 \\ 0 & \text{End}_k(K) \end{pmatrix}$$

It follows from Corollary 7 that $\eta : A \rightarrow \mathcal{O}(\mathbf{M})$ is injective. However, it is not an isomorphism in this case.

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